SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATOR FOR A TWO-FERMION SYSTEM ON A LATTICE: EIGENVALUES AND RESONANCES

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Abstract. *We investigate the Schrödinger operators $H_{\lambda\mu}(K)$, where K is the fixed quasi-momentum of a pair of identical fermions on a one-dimensional lattice $\mathbb{Z}$. The system has both nearest-neighbor and next-to-nearest neighbor interactions with parameters $\lambda\in\mathbb{R}$ and $\mu\in\mathbb{R}$, respectively. We partition the (λ,μ)-plane into three connected components, $\mathbb{G}_0$, $\mathbb{G}_1$, and $\mathbb{G}_2$. Each component of $H_{\lambda\mu}(0)$ possesses 0, 1, or 2 eigenvalues above its essential spectrum’s upper bound. We show that the upper threshold of $H_{\lambda\mu}(0)$ becomes a super-threshold resonance when the point (λ,μ) lies on the boundaries between these components, denoted by Γ_0 and Γ_1.*

Keywords: *Two-fermion system, discrete Schrödinger operator, essential spectrum, super-threshold resonance, the Lippmann-Schwinger operator.*

Panjaradagi ikki fermionli sistemaga mos shredinger operatorlarining spektral xususiyatlari: xosqiymatlar va rezonanslar

Annotatsiya. *Biz bir o‘lchovli panjara $\mathbb{Z}$ ustida joylashgan bir xil fermionlar juftining K kvazi-impulsi bilan bog‘liq $H_{\lambda\mu}(K)$ Shredinger operatorini o‘rganamiz. Tizimda eng yaqin qo‘shni va keyingi eng yaqin qo‘shnilar o‘rtasidagi o‘zaro ta’sirlar mavjud bo‘lib, ularning parametrlari mos ravishda $\lambda\in\mathbb{R}$ va $\mu\in\mathbb{R}$ bilan ifodalanadi. Biz (λ,μ) tekisligini uchta bog‘langan sohalar $\mathbb{G}_0$, $\mathbb{G}_1$ va $\mathbb{G}_2$ ga ajratamiz. $H_{\lambda\mu}(0)$ ning har bir komponenti uning asosiy spektrining yuqori qismida 0, 1 yoki 2 ta xos qiymatga ega. Biz shuni ko‘rsatdikki, (λ,μ) nuqta Γ_0 va Γ_1 bilan belgilangan ushbu sohalar chegarasida joylashganda, $H_{\lambda\mu}(0)$ ning yuqori chegara qiymati super-bo‘sag‘a rezonansiga aylanadi.*

Kalit so‘zlar: *Ikki fermionli tizim, diskret Shredinger operatori, asosiy spektr, super-bo‘sag‘a rezonansi, Lippmann-Shvinger operatori.*

Спектральные свойства операторов шрёдингера для двухфермионной системы на решётке: собственные значения и резонансы.

Аннотация. *Мы исследуем операторы Шрёдингера $H_{\lambda\mu}(K)$, где K — фиксированный квазиимпульс пары идентичных фермионов на одномерной решетке $\mathbb{Z}$. Система имеет как ближайшие, так и следующие за ближайшими соседние взаимодействия с параметрами $\lambda\in\mathbb{R}$ и $\mu\in\mathbb{R}$ соответственно. Мы разделяем плоскость (λ,μ) на три связные компоненты: $\mathbb{G}_0$, $\mathbb{G}_1$ и $\mathbb{G}_2$. Каждая компонента оператора $H_{\lambda\mu}(0)$ обладает 0, 1 или 2 собственными значениями, лежащими выше верхней границы её существенного* *спектра. Мы показываем, что верхний порог* $H_{\lambda\mu}(0)$ *становится суперпороговым резонансом, когда точка* (λ,μ) *лежит на границах между этими компонентами, обозначенными как* Γ_0 *и* Γ_1.

Ключевые слова: *Двухфермионная система, дискретный оператор Шрёдингера, существенный спектр, суперпороговый резонанс, оператор Липпмана-Швингера.*

1. Introduction

Lattice models are important in many areas of physics. They include the lattice two-body and three-body Hamiltonians, which are simplified versions of the Bose- and Fermi-Hubbard models respectively, concerning a constant number of a given particle species. It is worth noting that these two- and three-body lattice Hamiltonians have theoretically interesting in themselves. They have been studied in various papers, including references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Furthermore, these discrete Hamiltonians naturally approximate their continuous versions [12], enabling the study of few-body systems using bounded operator theory.

Unlike the continuous scenario, the two-body lattice system prevents decoupling of the center-of-mass movement. Nevertheless, discrete translation invariance enables the use of Floquet-Bloch decomposition (see, for example, [13, Section 4]). Specifically, in the quasimomentum representation, the total two-particle lattice Hamiltonian H for a system of identical fermions on a one-dimensional lattice $\mathbb{Z}$ with short-range interactions admits a von Neumann direct integral decomposition:

$$\mathrm{H} \simeq \int_{K\in T} \oplus\, H(K)dK, \tag{1}$$

where $\mathbb{T}$ denotes the one-dimensional torus and $H(K)$ is the fiber Hamiltonian acting on the corresponding functional Hilbert space over $\mathbb{T}$. Note that $H(K)$, the fiber Hamiltonians, have a non-trivial dependence on the quasimomentum, K, which belongs to the one-dimensional torus, $\mathbb{T}$. (See, for example, the references [14], [15].)

To gain more detailed information, we will focus on a two-term particle interaction. The first term is non-zero for nearest-neighbor particles. Similarly, the second term is significant only when particles are next-to-nearest neighbors (see Section 2.1 and Definition (8)). The terms contain real coupling constants λ and μ. The interaction operator in (quasi)momentum representation is denoted $V_{\lambda\mu}$.

In this paper, as the entries $H(K)$ in (1), we examine the collection of fiber operators

$$H_{\lambda\mu}(K) := H_0(K) + V_{\lambda\mu}, \qquad K \in \mathbb{T}, \tag{2}$$

where $H_0(K)$ is the fiber kinetic-energy operator,

$$(H_0(K)f)(p) = \mathcal{E}_K(p)f(p), \quad p \in \mathbb{T}, \quad f \in L^{2,o}(\mathbb{T}),$$

with

$$\mathcal{E}_K(p) := 4\cos\frac{K}{2}\cos p. \tag{3}$$

The interaction $V_{\lambda\mu}$ is an integral operator on $L^{2,o}(\mathbb{T})$ with a smooth kernel defined by (12). For any $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, formula (12) indicates that the rank of $V_{\lambda\mu}$ is 2.

Note that $V_{\lambda\mu}$ is independent of K. For $\lambda, \mu \in \mathbb{R}$, $H_0(K)$ and $V_{\lambda\mu}$ are bounded and self-adjoint operators. Due to the finite rank of $V_{\lambda\mu}$, the essential spectrum of $H_{\lambda\mu}(K)$ is the same as the spectrum of $H_0(K)$, i.e., it is the segment $[\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$, where

$$\mathcal{E}_{\min}(K) := -4\cos\frac{K}{2}, \qquad \mathcal{E}_{\max}(K) := 4\cos\frac{K}{2}.$$

In this model, we first found the precise number and location of eigenvalues for a given pair of interaction parameters, λ and μ, of the edge operator, $H_{\lambda\mu}(0)$.

This study primarily examines how variations in λ and μ cause eigenvalues of $H_{\lambda\mu}(K)$ to bifurcate from the essential spectrum, while also analyzing the inverse phenomenon where eigenvalues merge back into the essential spectrum.

We employ Fredholm determinants as a technical tool to reach this goal. Specifically, we consider the Fredholm determinant $\Delta_{\lambda\mu}(K,z)$ linked to the Lippmann-Schwinger operator formed by $H_0(K)$ and the perturbation $V_{\lambda\mu}$. For any fixed $K \in \mathbb{T}$, the eigenvalues of $H_{\lambda\mu}(K) = H_0(K) + V_{\lambda\mu}$ are in one-to-one correspondence with the zeros of $\Delta_{\lambda\mu}(K,z)$ (see [16], [17]).

We begin by examining the properties of the Fredholm determinant $\Delta_{\lambda\mu}(z) = \Delta_{\lambda\mu}(0,z)$ for the $K = 0$ case. We define $C_0(\lambda,\mu)$ as the leading term of the asymptotic expansion of $\Delta_{\lambda\mu}(z)$ as z nears the upper bound of the essential spectrum.

We prove that the zero of $\Delta_{\lambda\mu}(z)$ either emerges from the upper edge of $H_{\lambda\mu}(0)$ essential spectrum or merges into it, if and only if $C_0(\lambda,\mu) = 0$ vanishes (see Theorem 4.4). Further, we show that the curves Γ_0 and Γ_1, defined by the equation $C_0(\lambda,\mu) = 0$, divide the (λ,μ)-plane into three connected components, $\mathbb{G}_0, \mathbb{G}_1,$ and $\mathbb{G}_2$. We prove that in each component, $\mathbb{G}_j, j = 0,1,$ or 2, $H_{\lambda\mu}(0)$ has a fixed number of isolated eigenvalues, specifically j. In addition, in the combined sets $\mathbb{G}_0 \cup \Gamma_0$ and $\mathbb{G}_1 \cup \Gamma_1$, The count of eigenvalues for $H_{\lambda\mu}(0)$ also stays the same. Moreover, we establish that for each $(\lambda,\mu) \in \Gamma_0$ and $(\lambda,\mu) \in \Gamma_1$, the essential spectrum of $H_{\lambda\mu}(0)$ starts at $\mathcal{E}_{\max}(0)$, which corresponds to a super-threshold resonance. Furthermore, we have found that the number of eigenvalues of $H_{\lambda\mu}(0)$ changes by 1 if the point (λ,μ) in the parameter plane $\mathbb{R}^2$ crosses one of the boundaries Γ_0 and Γ_1 (see Figure 1(b) and Theorem 4.4).

Following the authors of papers [18] and [19], we refer to these boundary curves Γ_0 and Γ_1, as a *coupling curve thresholds* of the operator $H_{\lambda\mu}(0)$. Thus, we carefully examine the dynamics of the emergence or disappearance of eigenvalues as the values of the parameters (λ,μ) move from the origin $(0,0)$ towards infinity by crossing the boundaries of the connected components $\mathbb{G}_j$ on the parameter plane $\mathbb{R}^2$, where $j = 0,1,2$. These phenomena are described in Theorem 4.4.

In publications [10, 13, 20, 21] the two-particle Schrödinger operator $H_{\lambda\mu}(K)$, where $K \in \mathbb{T}^2$ is the quasimomentum of the two particles, is considered. These operators are associated with the Bose-Hubbard model, which describes a system of two identical particles (bosons or fermions) on the lattice $\mathbb{Z}^d$ with spatial dimension $(1 \le d \le 3)$. In these papers interactions between the particles are determined by two parameters: $\lambda, \mu \in \mathbb{R}$, which represent the strength magnitud of interactions between particles on one site and the nearest neighbour sites, respectively.

2. Hamiltonian of a lattice two-fermion system

2.1 The two-fermion Hamiltonian in the position–space representation

Let $\mathbb{Z}$ be the one-dimensional lattice and $l^{2,\mathrm{a}}(\mathbb{Z}\times\mathbb{Z}) \subset l^2(\mathbb{Z}\times\mathbb{Z})$, the Hilbert space of square–summable antisymmetric functions on $\mathbb{Z}\times\mathbb{Z}$.

In the position-space representation, the Hamiltonian $\hat{\mathbb{H}}_{\lambda\mu}$ associated with a system of two fermions with a nearest-neighbor and next-to-nearest-neighbor interactions potential $\hat{\mathbb{V}}_{\lambda\mu}$ is an operator on $l^{2,a}(\mathbb{Z}\times\mathbb{Z})$ of the following form:

$$\hat{\mathbb{H}}_{\lambda\mu}=\hat{\mathbb{H}}_0+\hat{\mathbb{V}}_{\lambda\mu},\ \lambda,\mu\in\mathbb{R}. \tag{4}$$

Here, $\hat{\mathbb{H}}_0$ is the kinetic energy operator of the system, defined on $l^{2,a}(\mathbb{Z}\times\mathbb{Z})$ as

$$[\hat{\mathbb{H}}_0\ \hat{f}](x,y)=\sum_{m\in\mathbb{Z}}\hat{\varepsilon}(x-m)\hat{f}(m,y)+\sum_{n\in\mathbb{Z}}\hat{\varepsilon}(y-n)\hat{f}(x,n),\ \hat{f}\in l^{2,a}(\mathbb{Z}\times\mathbb{Z}), \tag{5}$$

where

$$\hat{\varepsilon}(x)=\begin{cases}1, & |x|=1,\\ 0, & |x|\neq 1,\end{cases} \tag{6}$$

where $x\in\mathbb{Z}$. The first and second nearest-neighboring-site interaction potential $\hat{\mathbb{V}}_{\lambda\mu}$ is the operator of multiplication by a function $\hat{v}_{\lambda\mu}$,

$$[\hat{\mathbb{V}}_{\lambda\mu}\hat{f}](x,y)=\hat{v}_{\lambda\mu}(x-y)\hat{f}(x,y),\ \hat{f}\in l^{2,a}(\mathbb{Z}\times\mathbb{Z}), \tag{7}$$

where

$$\hat{v}_{\lambda\mu}(x)=\begin{cases}\lambda, & |x|=1,\\ \mu, & |x|=2,\\ 0, & in\quad other\quad cases.\end{cases} \tag{8}$$

Obviously, all the three operators $\hat{\mathbb{H}}_0$, $\hat{\mathbb{V}}_{\lambda\mu}$, and $\hat{\mathbb{H}}_{\lambda\mu}$ (for $\lambda,\mu\in\mathbb{R}$) are bounded and self-adjoint.

2.2 The two-fermion Hamiltonian in the quasimomentum representation

Let $\mathbb{T}$ be the one-dimensional torus, $\mathbb{T}=(\mathbb{R}/2\pi\mathbb{Z})\equiv[-\pi,\pi)$. The torus $\mathbb{T}$ represents the Pontryagin dual group of $\mathbb{Z}$, equipped with the Haar measure $\mathrm{d}p$. Let $L^{2,a}(\mathbb{T}\times\mathbb{T})$ be the Hilbert space of square-integrable antisymmetric functions on $\mathbb{T}\times\mathbb{T}$.

The quasimomentum-space version of the Hamiltonian (4) reads as

$$\mathbb{H}_{\lambda\mu}:=(\mathcal{F}\otimes\mathcal{F})\hat{\mathbb{H}}_{\lambda\mu}(\mathcal{F}\otimes\mathcal{F})^*,$$

where $\mathcal{F}\otimes\mathcal{F}$ denotes the Fourier transform. The operator $\mathbb{H}_{\lambda\mu}$ acts on $L^{2,a}(\mathbb{T}\times\mathbb{T})$ and has the form $\mathbb{H}_{\lambda\mu}=\mathbb{H}_0+\mathbb{V}_{\lambda\mu}$, where $\mathbb{H}_0=(\mathcal{F}\otimes\mathcal{F})\hat{\mathbb{H}}_0(\mathcal{F}\otimes\mathcal{F})^*$ is the multiplication operator:

$$[\mathbb{H}_0 f](p,q)=[\varepsilon(p)+\varepsilon(q)]f(p,q),$$

with

$$\varepsilon(p):=2\cos p,\quad p\in\mathbb{T},$$

the *dispersion relation* of a single fermion. The interaction $\mathbb{V}_{\lambda\mu}=(\mathcal{F}\otimes\mathcal{F})\hat{\mathbb{V}}_{\lambda\mu}(\mathcal{F}\otimes\mathcal{F})^*$ is the integral operator

$$[\mathbb{V}_{\lambda\mu}f](p,q)=\frac{1}{\pi}\int_{\mathbb{T}}v_{\lambda\mu}(p-u)f(u,p+q-u)\mathrm{d}u$$

with the kernel function

$$v_{\lambda\mu}(p)=\lambda\cos p+\mu\cos 2p$$

2.3 The Floquet-Bloch decomposition of $\mathbb{H}_{\lambda\mu}$ and discrete Schrödinger operators

Since $\hat{\mathbb{H}}_{\lambda\mu}$ commutes with the representation of the discrete group $\mathbb{Z}$ by shift operators on the lattice, the space $L^{2,a}(\mathbb{T}\times\mathbb{T})$ and $\mathbb{H}_{\lambda\mu}$ can be decomposed into the von Neumann direct integral as (see, e.g., [14])

$$L^{2,\mathrm{a}}(\mathbb{T}\times\mathbb{T})\simeq\int_{K\in\mathbb{T}}^{\oplus}L^{2,\mathrm{o}}(\mathbb{T})\mathrm{d}K \tag{9}$$

and

$$\mathbb{H}_{\lambda\mu}\simeq\int_{K\in\mathbb{T}}^{\oplus}H_{\lambda\mu}(K)\mathrm{d}K, \tag{10}$$

where $L^{2,\mathrm{o}}(\mathbb{T})$ is the Hilbert space of odd functions on $\mathbb{T}$. The fiber operator $H_{\lambda\mu}(K),\ K\in\mathbb{T}$, in (10) acting on $L^{2,\mathrm{o}}(\mathbb{T})$ is of the form

$$H_{\lambda\mu}(K):=H_0(K)+V_{\lambda\mu}, \tag{11}$$

where the (unperturbed) operator $H_0(K)$ is the multiplication operator by the function (3) and the perturbation operator $V_{\lambda\mu}$ is given by

$$[V_{\lambda\mu}f](p)=\frac{\lambda}{\pi}\sin p\int_{\mathbb{T}}\sin tf(t)\mathrm{d}t+\frac{\mu}{\pi}\sin 2p\int_{\mathbb{T}}\sin 2tf(t)\mathrm{d}t. \tag{12}$$

Obviously, both the operators $H_0(K)$ and $V_{\lambda\mu}$ are bounded and self-adjoint. In the literature, the parameter $K\in\mathbb{T}$ is called the *two-particle quasimomentum* and the entry $H_{\lambda\mu}(K)$ is called the *discrete Schrödinger operator* associated to the two-particle Hamiltonian $\hat{\mathbb{H}}_{\lambda\mu}$.

2.4 The essential spectrum of discrete Schrödinger operators

Depending on $\lambda,\mu\in\mathbb{R}$, the rank of $V_{\lambda\mu}$ varies but never exceeds two. Hence, by Weyl's theorem, for any $K\in\mathbb{T}$ the essential spectrum $\sigma_{\mathrm{ess}}(H_{\lambda\mu}(K))$ of $H_{\lambda\mu}(K)$ coincides with the spectrum of $H_0(K)$, i.e.,

$$\sigma_{\mathrm{ess}}(H_{\lambda\mu}(K))=\sigma(H_0(K))=[\mathcal{E}_{\min}(K),\mathcal{E}_{\max}(K)], \tag{13}$$

with

$$\mathcal{E}_{\min}(K):=\min_{p\in\mathbb{T}}\mathcal{E}_K(p)=-4\cos\frac{K}{2}\geq\mathcal{E}_{\min}(0)=-4,\quad \mathcal{E}_{\max}(K):=\max_{p\in\mathbb{T}}\mathcal{E}_K(p)=4\cos\frac{K}{2}\leq\mathcal{E}_{\max}(0)=4.$$

3. Auxiliary statements

3.1 The Lippmann–Schwinger operator

Let $\{\alpha_i, i=1,2\}$ be a system of vectors in $L^{2,o}(\mathbb{T})$, with

$$\alpha_1(p)=\frac{\sin p}{\sqrt{\pi}},\alpha_2(p)=\frac{\sin 2p}{\sqrt{\pi}} \tag{15}$$

One easily verifies by inspection that the vectors (15) are orthonormal in $L^{2,o}(\mathbb{T})$. By using the orthonormal systems (15) obtains

$$V_{\lambda\mu}f=\lambda(f,\alpha_1)\alpha_1+\mu(f,\alpha_2)\alpha_2 \tag{16}$$

where $(\cdot,\cdot)$ is the inner product in $L^{2,0}(\mathbb{T})$. For any $z\in\mathbb{C}\setminus[-4,4]$ we define (the transpose of) the Lippmann-Schwinger operator (see., e.g., [22]) as

$$B_{\lambda\mu}(0,z)=-V_{\lambda\mu}R_0(0,z),$$

where $R_0(0,z):=[H_0(0)-zI]^{-1}, z\in\mathbb{C}\setminus[0,4]$, is the resolvent of the operator $H_0(0)$.

Lemma 3.1 *For each* $\lambda,\mu\in\mathbb{R}$ *the number* $z\in\mathbb{C}\setminus[-4,4]$ *is an eigenvalue of the operator* $H_{\lambda\mu}(0)$ *if and only if the number* 1 *is an eigenvalue for* $B_{\lambda\mu}(0,z)$.

The proof of this lemma is quite standard (see., e.g., [16]) and, thus, we omit it.

The representation (16) yields the equivalence of the Lippmann-Schwinger equation

$$B_{\lambda\mu}(0,z)\varphi=\varphi,\ \varphi\in L^{2,o}(\mathbb{T}) \tag{17}$$

to the following algebraic linear system in $x_i:=(\varphi,\alpha_i), i=1,2$:

$$\begin{cases} (1+\lambda a(z))x_1 + \lambda c(z)x_2 = 0, \\ \mu c(z)x_1 + (1+\mu b(z))x_2 = 0. \end{cases} \tag{18}$$

where

$$a(z) = \frac{1}{\pi}\int_{\mathbb{T}} \frac{\sin^2 p\mathrm{d}p}{\mathcal{E}_0(p)-z}, \tag{19}$$

$$b(z) = \frac{1}{\pi}\int_{\mathbb{T}} \frac{\sin^2 2p\mathrm{d}p}{\mathcal{E}_0(p)-z}, \tag{20}$$

$$c(z) = \frac{1}{\pi}\int_{\mathbb{T}} \frac{\sin p \sin 2p\mathrm{d}p}{\mathcal{E}_0(p)-z}. \tag{21}$$

Thus we write

$$\Delta_{\lambda\mu}(z) := \det[I - B_{\lambda\mu}(0,z)], z \in \mathbb{C}\setminus[-4,4].$$

Lemma 3.2 *A number* $z \in \mathbb{C}\setminus[-4,4]$ *is an eigenvalue of the operator* $H_{\lambda\mu}(0)$ *if and only if*

$$\Delta_{\lambda\mu}(z) = 0. \tag{22}$$

The proof of this lemma is quite standard (cf., e.g., [20, 21])

Lemma 3.3 *For any* $\lambda,\mu \in \mathbb{R}$ *the determinant* $\Delta_{\lambda\mu}(z)$ *has the form*

$$\Delta_{\lambda\mu}(z) = \Delta_{\lambda 0}(z)\Delta_{0\mu}(z) - \lambda\mu c^2(z), \tag{23}$$

where

$$\Delta_{\lambda 0}(z) = 1 + \lambda a(z), \tag{24}$$

$$\Delta_{0\mu}(z) = 1 + \mu b(z), \tag{25}$$

Proof. Direct computation of the determinant gives the result.

Lemma 3.4 *The functions* $a(z), b(z)$ *and* $c(z)$ *defined in* $\mathbb{R}\setminus[-4,4]$ *are real-valued and, moreover, they are strictly increasing in* $(4,+\infty)$*, with the following asymptotics:*

$$a(z) = -\frac{1}{2} + \frac{\sqrt{2}}{4}\sqrt{z-4} + o(\sqrt{z-4}),\ z \to 4^+, \tag{26}$$

$$b(z) = -1 + \sqrt{2}\sqrt{z-4} + o(\sqrt{z-4}),\ z \to 4^+, \tag{27}$$

$$c(z) = -\frac{1}{2} + \frac{\sqrt{2}}{2}\sqrt{z-4} + o(\sqrt{z-4}),\ z \to 4^+, \tag{28}$$

where $\sqrt{z-4}$ denotes the branch of the corresponding analytic function that is real for $z > 4$.

Proof. The proof of this lemma is similar to that of Lemma 4.4 in [10].

Lemma 3.5 *The function* $\Delta_{\lambda\mu}(z)$ *is real-valued in* $\mathbb{R}\setminus[-4,4]$ *and has the following asymptotics:*

$$\Delta_{\lambda\mu}(z) = C_0(\lambda,\mu) + C_1(\lambda,\mu)\sqrt{z-4} + o(\sqrt{z-4}),\ z \to 4^+, \tag{29}$$

where

$$C_0(\lambda,\mu) = \frac{1}{4}\lambda\mu - \mu - \frac{1}{2}\lambda + 1 \tag{30}$$

and

$$C_1(\lambda,\mu) = -\frac{\sqrt{2}}{4}\lambda\mu + \sqrt{2}\mu + \frac{\sqrt{2}}{4}\lambda.$$

Proof. Given Lemmas 3.3 and 3.4, the proof can be obtained by direct calculation.

Next, we will study the number and location of the roots of the functions $\Delta_{\lambda 0}$ and $\Delta_{0\mu}$, which are defined in (24) and (25).

Lemma 3.6 *Let* $\lambda \in \mathbb{R}$ *and let* $\lambda_0 = 2$. *Then:*

(i) If $\lambda < \lambda_0$, *then* $\Delta_{\lambda 0}(\cdot)$ *has no roots in* $(4,+\infty)$.

(ii) If $\lambda > \lambda_0$, *then* $\Delta_{\lambda 0}(\cdot)$ *has a unique root* $\zeta(\lambda)$ *in* $(4,+\infty)$.

Lemma 3.7 *Let* $\mu \in \mathbb{R}$ *and let* μ_{01} *be as in (31). Then:*

(i) If $\mu < \mu_{01}$, *then* $\Delta_{0\mu}(\cdot)$ *has no roots in* $(4,+\infty)$.

(ii) If $\mu > \mu_{01}$, *then* $\Delta_{0\mu}(\cdot)$ *has a unique root* $\eta(\mu)$ *in* $(4,+\infty)$.

Proof. Both Lemmas 3.6 and 3.7 are proven by using the representations (24) and (25) from Lemma 3.3 and the asymptotical formulae for $a(z)$, $b(z)$ and $c(z)$ from Lemma 3.4.

4 Main results

4.1 The threshold resonances of the Schrödinger operator

Definition 4.1 *Let the equation* $H_{\lambda\mu}(0)f = \mathcal{E}_{\max}(0)f$ *has a nonzero solution* $f \in L^{\varepsilon,o}(\mathbb{T}) \setminus L^{1,o}(\mathbb{T})$. *Then the upper threshold* $\mathcal{E}_{\max}(0)$ *is called super-threshold resonance.*

We introduce the notations:

$$\mu_{01} = 1, \quad \mu_{02} = 2 \quad \text{and} \quad a_0 = 4. \tag{31}$$

Hence, the numbers μ_{01} and μ_{02} satisfy the relations

$$0 < \mu_{01} < \mu_{02}. \tag{32}$$

By using the numbers μ_{01}, μ_{02} and a_0, defined in (31), the function $C_0(\lambda,\mu)$ can be written in the following forms:

$$C_0(\lambda,\mu) = \frac{1}{4}[-a_0(\mu-\mu_{01}) + \lambda(\mu-\mu_{02})]. \tag{33}$$

Lemma 4.2 *The set of points satisfying the equation* $C_0(\lambda,\mu) = 0$ *coincides with the graph of function* $\lambda(\mu)$:

$$\lambda(\mu) = a_0\frac{\mu-\mu_{01}}{\mu-\mu_{02}}, \mu \in \mathbb{R} \setminus \{\mu_{02}\}.$$

Proof. Now we prove that $C_0(\lambda,\mu) = 0$ implies $\mu - \mu_{02} \neq 0$, i.e., the following system of equations has no solution

$$\begin{cases} C_0(\lambda,\mu) = 0 \\ \mu - \mu_{02} = 0 \end{cases} \tag{34}$$

Using the representation (33) and taking into account the second equality in (34) we obtain the following equivalent system

$$\begin{cases} \mu - \mu_{01} = 0 \\ \mu - \mu_{02} = 0 \end{cases}$$

since by (32) the numbers μ_{01} and μ_{02} are different from each other, this system has no solution. Thus $C_0(\lambda,\mu) = 0$ implies $\mu - \mu_{02} \neq 0$, and vice versa.

We also introduce the following notations:

$$I_0 = (-\infty, \mu_{02}), \quad I_1 = (\mu_{02}, +\infty).$$

The straight line

$$\mu = \mu_{02}$$

divides the graph of the function $\lambda(\mu)$ into two continuous curves (Γ_0, Γ_1):

$$\Gamma_j = \{(\lambda,\mu) \in \mathbb{R}^2 : \lambda(\mu) = a_0\frac{\mu-\mu_{01}}{\mu-\mu_{02}}, \mu \in I_j\}, j = 0,1.$$

The curves Γ_0 and Γ_1 divide the (λ,μ)-plane into three connected components $\mathbb{G}_0$, $\mathbb{G}_1$, $\mathbb{G}_2$

(see Fig 1(a)):

$$\mathbb{G}_0 := \{(\lambda,\mu)\in\mathbb{R}^2 : \lambda(\mu) < a_0\frac{\mu-\mu_{01}}{\mu-\mu_{02}}, \mu\in I_0\},$$

$$\mathbb{G}_1 := \{(\lambda,\mu)\in\mathbb{R}^2 : \lambda(\mu) > a_0\frac{\mu-\mu_{01}}{\mu-\mu_{02}}, \mu\in I_0\}\cup\{(\lambda,\mu)\in\mathbb{R}^2 : \lambda(\mu) < a_0\frac{\mu-\mu_{01}}{\mu-\mu_{02}}, \mu\in I_1\},$$

$$\mathbb{G}_2 := \{(\lambda,\mu)\in\mathbb{R}^2 : \lambda(\mu) > a_0\frac{\mu-\mu_{01}}{\mu-\mu_{02}}, \mu\in I_1\}$$

For further consideration, we will rewrite the coefficient $C_1(\lambda,\mu)$ in the same way as we did for $C_0(\lambda,\mu)$ in equation (33):

$$C_1(\lambda,\mu) = -\frac{\sqrt{2}}{4}[-a_1(\mu-\mu_{11})+\lambda(\mu-\mu_{12})],$$

where

$$\mu_{11}=0,\quad \mu_{12}=1 \quad \text{and} \quad a_1=4,$$

and satisfy the inequalities

$$0=\mu_{11}<\mu_{12}=\mu_{01}<\mu_{02},\quad 0<a_1=a_0. \tag{35}$$

The curve on the (λ,μ)-plane, defined by the equation $C_1(\lambda,\mu)=0$ coincides with the graph of the function $\lambda(\mu)=a_1\frac{\mu-\mu_{11}}{\mu-\mu_{12}}$, which consists of hyperbolas with asymptotes at $\mu=\mu_{12}$. These curves do not intersect the curves $C_0(\lambda,\mu)=0$ and partition the plane $\mathbb{R}^2$ into three connected components (see Appendix A).

Theorem 4.3 *Let* $\mathbb{G}$ *be one of the above connected components* $\mathbb{G}_0$, $\mathbb{G}_1$ *and* $\mathbb{G}_2$. *If* $(\lambda,\mu)\in\mathbb{G}$, *the number* $n_-(H_{\lambda\mu}(0))$ *of eigenvalues of* $H_{\lambda\mu}(0)$ *lying above the essential spectrum* $\sigma_{\text{ess}}(H_{\lambda\mu}(0))$, *remains constant.*

Proof. The proof of Theorem 4.3 is analogous to proof of Theorem 3.2 (see, [13]).

Theorem 4.4 *For any* $\lambda,\mu\in\mathbb{R}$, *the number of eigenvalues of* $H_{\lambda\mu}(0)$ *lying above the essential spectrum, as well as the super-threshold resonance of* $H_{\lambda\mu}(0)$, *can be described as follows:*

(i) For any $(\lambda,\mu)\in\mathbb{G}_0$, *the operator* $H_{\lambda\mu}(0)$ *has no eigenvalues above the essential spectrum.*

(ii) For any $(\lambda,\mu)\in\Gamma_0$, *the operator* $H_{\lambda\mu}(0)$ *has no eigenvalues above the essential spectrum. But* $\mathcal{E}_{\max}(0)$ *is a upper super-threshold resonance of* $H_{\lambda\mu}(0)$. *The corresponding resonance functions are of the form:*

$$f_{\lambda\mu}(p) = \frac{(\sin p + \frac{2-\lambda}{\lambda}\sin 2p)C}{\mathcal{E}_0(p)-\mathcal{E}_{\max}(0)}.$$

Here, the constant C *is a nonzero real number.*

(iii) For any $(\lambda,\mu)\in\mathbb{G}_1$, *the operator* $H_{\lambda\mu}(0)$ *has one eigenvalue* $z_1(\lambda,\mu;0)$ *above the essential spectrum, and the associated eigenfunctions are of the form:*

$$f_{\lambda\mu}(p) = \frac{(\sin p + k(z_1(\lambda,\mu,0))\sin 2p)C}{\mathcal{E}_0(p)-z_1(\lambda,\mu;0)}, \tag{36}$$

where C *is a nonzero real number and* $k(z_1)=\frac{\Delta_{\lambda 0}(z_1)}{\lambda c(z_1)}$.

(iv) For any $(\lambda,\mu)\in\Gamma_1$*, the operator* $H_{\lambda\mu}(0)$ *has one eigenvalue* $z_1(\lambda,\mu;0)$ *above the essential spectrum. Moreover,* $\mathcal{E}_{\max}(0)$ *is a upper super-threshold resonance of* $H_{\lambda\mu}(0)$.

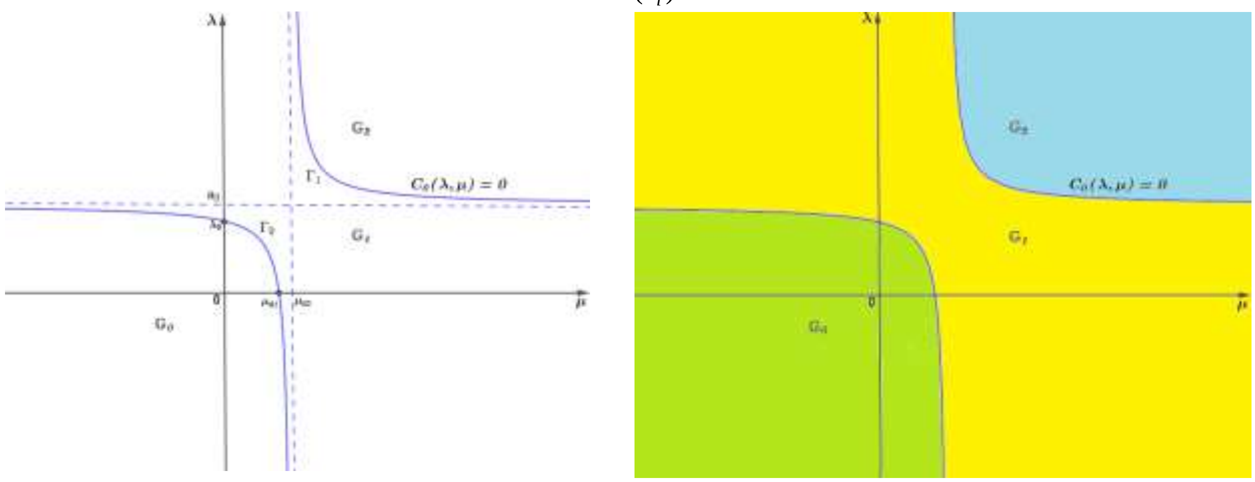
(v) For any $(\lambda,\mu)\in\mathbb{G}_2$*, the operator* $H_{\lambda\mu}(0)$ *has exactly two eigenvalues,* $z_1(\lambda,\mu;0)$ *and* $z_2(\lambda,\mu;0)$*. These eigenvalues satisfy the inequalities*

$$4 < z_2(\lambda,\mu;0) < z_1(\lambda,\mu;0),$$

and the associated eigenfunctions are of the form:

$$f^i_{\lambda\mu}(p)=\frac{(\sin p + k(z_i(\lambda,\mu,0))\sin 2p)C}{\mathcal{E}_0(p)-z_i(\lambda,\mu;0)}, \tag{37}$$

where C *is a nonzero real number and* $k(z_i)=\dfrac{\Delta_{\lambda 0}(z_i)}{\lambda c(z_i)}, i=1,2.$



(a) (b)

FIGURE 1. (*a*) The connected components $\mathbb{G}_0$, $\mathbb{G}_1$, $\mathbb{G}_2$ and their boundaries, (*b*) Partition of the (λ,μ)-plane of parameters $\lambda,\mu\in\mathbb{R}$ in the connected components $\mathbb{G}_\alpha, \alpha=0,1,2$ (see Theorem 4.4).

5. Proofs of the main results

In this section we prove our main result, Theorem 4.4.

Proof of Theorem 4.4. We only prove items (iii), (iv) and (v). The remaining items can be proven similarly.

(iii) Assume $(\lambda,\mu)\in\mathbb{G}_1$. By Lemma 3.7, for any $\mu>\mu_{01}$ the operator $H_{0\mu}$ has unique eigenvalue in $(4,+\infty)$ at the point $(0,\mu)\in\mathbb{G}_1$. Then by Theorem 4.3 for any $(\lambda,\mu)\in\mathbb{G}_1$ the operator $H_{\lambda\mu}(0)$ has unique eigenvalue in $(4,+\infty)$.

(iv) If $(\lambda,\mu)\in\Gamma_1$, then the item (ii) of Lemma 6.1 yields that

$$C_1(\lambda,\mu)<0.$$

Since the function $f(z)=\sqrt{z-4}$ is real-valued, positive, and strictly increasing on the interval $(4,+\infty)$, it follows that

$$C_1(\lambda,\mu)\sqrt{z-4}<0, \quad z\in(4,+\infty).$$

Thus, there exists a number δ such that

$$\Delta_{\lambda\mu}(z)<0, \quad z\in(4,4+\delta)$$

where $\delta>0$ is a sufficiently small number.

Additionally, we have

$$\lim_{z\to+\infty}\Delta_{\lambda\mu}(z)=1.$$

Therefore, the function $\Delta_{\lambda\mu}(z)$ has only one root $z_1(\lambda,\mu;0)$ satisfying

$$4<z_1(\lambda,\mu;0).$$

Otherwise it would have at least three roots in $(4,+\infty)$, but this is impossible.

Hence, Lemma 3.2 implies that the operator $H_{\lambda\mu}(0)$ has unique eigenvalue above the essential spectrum.

We will prove that $\mathcal{E}_{\max}(0)$ is a super-threshold resonance of $H_{\lambda\mu}(0)$.

By Lemma 3.1, the operator $H_{\lambda\mu}(0)$ has eigenvalue $z\in\mathbb{C}\setminus[-4,4]$ if and only if the homogeneous equation (17) has a nonzero solution $\varphi\in L^{2,o}(\mathbb{T})$, and the associated eigenfunction, given by

$$f_{\lambda\mu}=R_0(0,z)\varphi \tag{38}$$

belongs to $L^{2,o}(\mathbb{T})$.

From (18) by the asymptotics of the functions $a(z),b(z),c(z)$ $z\to\ \mathcal{E}_{\max}(0)^+$ yields that

$$\begin{cases}(1-\frac{1}{2}\lambda)C_1-\frac{1}{2}\lambda C_2=0,\\ -\frac{1}{2}\mu C_1+(1-\mu)C_2=0.\end{cases} \tag{39}$$

Due to the Lemma 3.5, we obtain that

$$\Delta_{\lambda\mu}(\mathcal{E}_{\max}(0))=C_0(\lambda,\mu).$$

We have $C_0(\lambda,\mu)=0$. From system (39), we express one of the unknown coefficients in terms of the other:

$$C_2=\frac{2-\lambda}{\lambda}C_1.$$

Then, the function $\varphi(p)$ can be rewritten as

$$\varphi(p)=(\sin p+\frac{2-\lambda}{\lambda}\sin 2p)C_1$$

where C_1 is a nonzero real number.

The equation (38) gives

$$f_{\lambda\mu}(p)=f_\lambda(p)=\frac{(\sin p+\frac{2-\lambda}{\lambda}\sin 2p)C}{\mathcal{E}_0(p)-\mathcal{E}_{\max}(0)}$$

Since the functions $\varphi(p)$ and $(\mathcal{E}_0(p)-\mathcal{E}_{\max}(0))$ behave like $|p|$ and $|p|^2$ near the point 0, for any $\varepsilon\in(0,1)$, we can conclude

$$\int_{\mathbb{T}}|f_{\lambda\mu}(p)|^{\varepsilon}\,dp=\lim_{z\to\mathcal{E}_{\max}(0)}\int_{\mathbb{T}}\frac{|\varphi(p)|^{\varepsilon}\,dp}{|\mathcal{E}_0(p)-z|^{\varepsilon}}<+\infty,$$

and

$$\int_{\mathbb{T}}|f_{\lambda\mu}(p)|\,dp=\lim_{z\to\ \mathcal{E}_{\max}(0)}\int_{\mathbb{T}}\frac{|\varphi(p)|\,dp}{\mathcal{E}_0(p)-z}=+\infty.$$

Therefore, $f_{\lambda\mu}\in L^{\varepsilon,o}(\mathbb{T})$, but $f_{\lambda\mu}\notin L^{1,o}(\mathbb{T})$.

(v) Let $(\lambda,\mu)\in\mathbb{G}_2$. By the definition of the set $\mathbb{G}_2$, the following inequality holds:

$$C_0(\lambda,\mu)>0.$$

Consequently, Lemma 3.5 implies:

$$\lim_{z\to 4} \Delta_{\lambda\mu}(z) > 0. \tag{40}$$

Using the Lebesgue Dominated Convergence Theorem, we also have:

$$\lim_{z\to +\infty} \Delta_{\lambda\mu}(z) = 1. \tag{41}$$

By Lemmas (3.6) and (3.7), for any $\lambda > \lambda_0$ and $\mu > \mu_{01}$, the functions $\Delta_{\lambda 0}(\cdot)$ and $\Delta_{0\mu}(\cdot)$ have unique zeros, denoted as $\zeta(\lambda)$ and $\eta(\mu)$, respectively. Moreover,

$$c^2(\zeta(\lambda)) > 0 \quad \text{and} \quad c^2(\eta(\mu)) > 0.$$

In this case, $\Delta_{\lambda\mu}(\cdot)$ takes negative values either at $z_0 = \zeta(\lambda)$ or $z_0 = \eta(\mu)$. Using the asymptotics in (40) and (41) at $z = 4$ and at $z = +\infty$, and the continuity of $\Delta_{\lambda\mu}(\cdot)$, it follows that $\Delta_{\lambda\mu}(\cdot)$ has two zeros:

$$z_2(\lambda,\mu;0) \in (4, z_0) \quad \textit{and} \quad z_1(\lambda,\mu;0) \in (z_0, +\infty).$$

Thus, $\Delta_{\lambda\mu}(\cdot)$ has exactly two zeros in the interval $(4,+\infty)$.

By Lemma 3.2, the operator $H_{\lambda\mu}$ has exactly two eigenvalues above the essential spectrum.

Appendix A

Lemma 6.1 *Let* $\lambda, \mu \in \mathbb{R}$.

(i) For any $(\lambda,\mu) \in \Gamma_0$, *the inequality* $C_1(\lambda,\mu) > 0$ *holds.*

(ii) For any $(\lambda,\mu) \in \Gamma_1$, *the inequality* $C_1(\lambda,\mu) < 0$ *holds.*

Proof. (i) Let $(\lambda,\mu) \in \Gamma_0$, then $C_0(\lambda,\mu) = 0$. By definition Γ_0 and (35) the following inequalities

$$\mu - \mu_{02} < 0$$

hold. Therefore

$$C_1(\lambda,\mu) = -\frac{\sqrt{2}}{4}[\lambda(\mu-1) - 4\mu] = -\frac{\sqrt{2}}{4}[4\frac{\mu-1}{\mu-2}(\mu-1) - 4\mu] = -\frac{\sqrt{2}}{\mu-2} > 0.$$

(ii) Let $(\lambda,\mu) \in \Gamma_1$, then $C_0(\lambda,\mu) = 0$. By definition Γ_1 and (35) the following inequalities

$$\mu - \mu_{02} > 0 \tag{42}$$

hold. Similarly, using inequality (42), we obtain

$$C_1(\lambda,\mu) < 0.$$

Corollary. The curves defined by the equations $C_0(\lambda,\mu) = 0$ and $C_1(\lambda,\mu) = 0$ do not intersect.

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