Personalized Collatz sequence: residue analysis and uniqueness of the trivial cycle

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ABSTRACT

In this paper we analyze the Collatz sequence in the custom formulation I proposed in the previous articles and introduce the concept of residue (Res), defined as the ratio between the reduction and expansion factors in each composite iteration. It is shown that for every odd integer N, the value of always remains close to 1 and between 0.9 and 1.25. It follows that there are no other trivial cycles, except for the $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$.

KEYWORDS

Residue, iteration composed, personalized succession

1. Introduction

In this sixth article, the personalized succession equivalent to the canonical Collatz succession is presented again because it leads to the same final results. Subsequently, the term succession will indicate the custom sequence and the term iteration of the same will indicate an **iteration composed of a certain number of simple iterations of the type**:

1. Odd steps (application of $3N + 1$)
2. Even passes (division by 2)

depending on the initial number of the same iteration.

Connecting to my previous works in which the veracity of the conjecture is demonstrated, this article introduces the concept of **residue** (Res) with which it is demonstrated that in the Collatz sequence I personalized and in the canonical one there are no other trivial cycles besides the $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$ cycle.

2. Definition of residue

For each composite iteration of the sequence that leads to $N \rightarrow N'$ with:

a: number of odd passes (applications of $3n + 1$),

b: number of even passes (divisions by 2),

Residue is defined as:

$$\text{Res} = \frac{2^b \cdot N'}{3^a \cdot N}$$

which represents the correction factor of the sum of +1 in odd passages, from which we have

$$N' = \frac{3^a \cdot N \cdot Res}{2^b}$$

3. The four cases of personalized succession

Both the initial numbers N and the final numbers N' in each iteration of the sequence are all odd of the type ≡ 1 mod. 4 or ≡ 3 mod. 4 (after a possible initial number of the sequence which can also be even mod. 4) then each iteration reduces to these cases in which the value of Res :

1. **Case 1:**

$$N = 4(1+2k)+1 \Rightarrow N' = 1+2k$$

$$a = 0, b = 2$$

$$\text{Res} = \frac{4(1+2k)}{4(1+2k)+1} = 1 - \frac{1}{8k+5}.$$

Value < 1, tends to 1 for $k \to \infty$.

2. **Case 2:**

$$N = 8k+1 \Rightarrow N' = 6k+1$$

$$a = 1, b = 2$$

$$\text{Res} = \frac{4(6k+1)}{3(8k+1)} = 1 + \frac{1}{24k+3}.$$

Value > 1, tends to 1 for $k \to \infty$.

3. **Case 3 (p=2 and with k even):**

$$N = 2^2(1+2k)-1 \Rightarrow N' = \frac{3(1+2k)-1}{2}$$

$$a = 1, b = 3$$

$$\text{Res} = 1 - \frac{1}{24k+9}.$$

Value < 1, tends to 1 for even, and $k \to \infty$.

4. **Case 3 (p ≥3 and with k the same parity as p):**

$$N = 2^{\text{p}}(1+2k)-1 \Rightarrow N' = [3^{p-1}(1+2k)-1]/2$$

$$a = \text{p}-1, b = \text{p}+1$$

$$\text{Res} = \frac{2^{p+1}3^{p-1}k + 2^p(3^{p-1}-1)}{2^{p+1}3^{p-1}k + 3^{p-1}(2^p-1)} = 1 + \frac{3^{p-1}-2^p}{2^{p+1}3^{p-1}k + 3^{p-1}(2^p-1)}.$$

Value > 1, tends to 1 for $k \to \infty$ con k avente la stessa parità di p.

5. **Case 4 (p ≥2 and k different from p):**

$$N = 2^{\text{p}}(1+2k)-1 \Rightarrow N' = [3^p(1+2k)-1]/2$$

$$a = \text{p}, b = \text{p}+1$$

$$\text{Res} = \frac{2^{p+1}3^p k + 2^p(3^p-1)}{2^{p+1}3^p k + 3^p(2^p-1)} = 1 + \frac{3^p-2^p}{2^{p+1}3^p k + 3^p(2^p-1)}.$$

Value > 1, tends to 1 for $k \to \infty$ con k avente diversa parità di p.

4. Trivial cycle search under the previous four cases

We show that, imposing N′=N in each previous case, the only solution is N=1. In fact:

- Case 1:

Equations: N=4(1+2k)+1, N′=1+2k. Imposing N′=N would give 1+2k = 4(1+2k)+1 ⇒ 1+2k = 5+8k ⇒ 6k = −4 ⇒ k = −2/3, not integer. So no fixed point.

- Case 2:

Equations: N=8k+1, N′=6k+1. Imposing N′=N gives 8k+1 = 6k+1 ⇒ 2k = 0 ⇒ k=0 ⇒ N=1, N′=1. This is the fixed point (and coincides with the trivial cycle 1→4→2→1) with a=1 and b=2 and Res = 4/3.

- Case 3 (p≥2, same parity of k):

Equations: N=2^p(1+2k)−1, N′ = [3^{p−1}(1+2k)−1]/2.

Impose N′=N gate 2· (2^p(1+2k)−1) = 3^{p−1}(1+2k)−1, so (1+2k)· (3^{p−1} − 2^{p+1}) = −1.

The factor (1+2k) is a positive integer; the factor (3^{p−1} − 2^{p+1}) is an integer. For p=2 is 3−8=−5, which cannot give −1 when multiplied by an integer ≥1. For p≥3, 3^{p−1} − 2^{p+1} ≤ −1 or ≫ 0, but in any case it does not allow the product exactly −1 with (1+2k)≥1. So no solution.

- Case 4 (p≥2, different parity of k):

Equations: N=2^p(1+2k)−1, N′ = [3^{p}(1+2k)−1]/2.

Impose N′=N gate 2· (2^p(1+2k)−1) = 3^{p}(1+2k)−1, so (1+2k)· (3^{p} − 2^{p+1}) = −1.

With (1+2k)≥1, the product cannot be −1. So no solution.

Conclusion: therefore, within each iteration of the custom sequence, the only case that admits N′=N is Case 2 with k=0, i.e. N=1.

If there is an additional loop N-->N' with N’=N other than the trivial one, it must come out of a group of successive compound iterations.

5. Multiplicative property, proximity to 1 and growth of b with respect to a

Consider two consecutive compound iterations: the first brings $N \to N'$con a_1 odd steps and b_1 same; the second door $N' \to N''$ with a_2 odd steps and b_2 same. By definition

$$\mathrm{Res}_1 = \frac{2^{b_1} N'}{3^{a_1} N}, \mathrm{Res}_2 = \frac{2^{b_2} N''}{3^{a_2} N'}.$$

Then

$\mathrm{Res}_1 \cdot \mathrm{Res}_2 = \frac{2^{b_1+b_2} N''}{3^{a_1+a_2} N} = \mathrm{Res}_{12}$ for the block $N \to N''$,

i.e. the product of the Res of consecutive blocks is the Res of the aggregate block and is valid identically for any concatenation of blocks.

For a concatenation of iterations:

$$\mathrm{Res}_{\mathrm{tot}} = \prod_i \mathrm{Res}_i.$$

If each $\mathrm{Res}_i = 1 \mp \varepsilon_i$ with $|\ \varepsilon_i\ |$ small (of the order $\frac{1}{k_i}$), then

$$\log\ \mathrm{Res}_{\mathrm{tot}} = \sum_i \log\left(1+\varepsilon_i\right) \approx \sum_i \varepsilon_i,$$

so the sum of small corrections keeps $\mathrm{Res}_{\mathrm{tot}}$ close to 1, compatibly with the fact that the sequence alternates cases with $\mathrm{Res} < 1$ and $\mathrm{Res} > 1$.

In addition, in the sequence, each block assigns counts a, b such that:

- Case 1: $a = 0, b = 2 \Rightarrow b = a + 2$.
- Case 2: $a = 1, b = 2 \Rightarrow b = a + 1$.
- Case 3: $a = p - 1, b = p + 1 \Rightarrow b = a + 2$.
- Case 4: $a = p, b = p + 1 \Rightarrow b = a + 1$.

In all cases, $b \geq a + 1$. By aggregating blocks along the trajectory, it is systematically obtained $\sum b > \sum a \Rightarrow \sum b - \sum a \geq number\ of\ iterations$. This is the heart of the tendency to contraction in the 2-adic sense (more divisions by 2 than expansions by 3).

Table 1. Numerical example on N=27.

Customized Collatz sequence using only odd numbers N≡1 mod 4 and N≡3 mod 4

Res for each iteration	Res compared to initial N	Case of the customized succession	Iterations	step = Odd + Even	N	periodicity p or even and odd numbers	Value of 1+2*k	Value of k	odd step	even pari	Type of N and its transformation
			0		27						
1,02057613169	1,02057613169	case 4 with p>=2	1	5	31	2	7	3	2	3	4*h0+3-->3*h1+1
1,02801008894	1,04916255991	case 4 with p>=2	2	16	121	5	1	0	7	9	4*h0+3-->3*h1+1
1,00275482094	1,05205281489	case 2	3	19	91	1d2p			8	11	4*h0+1-->3*h0+1
1,00610500611	1,05847560375	case 4 with p>=2	4	24	103	2	23	11	10	14	4*h0+3-->3*h1+1
1,00683207479	1,06570718824	case 4 with p>=2	5	31	175	3	13	6	13	18	4*h0+3-->3*h1+1
1,00458553792	1,07059402897	case 4 with p>=2	6	40	445	4	11	5	17	23	4*h0+3-->3*h1+1
0,99775280899	1,06818819969	case 1	7	42	111	0d2p			17	25	4*h0+1-->h0
1,00722945167	1,07591061466	case 4 with p>=2	8	51	283	4	7	3	21	30	4*h0+3-->3*h1+1
1,00196309384	1,07802272815	case 4 with p>=2	9	56	319	2	71	35	23	33	4*h0+3-->3*h1+1
1,00230917089	1,08051206685	case 3 with p>=3	10	68	607	6	5	2	28	40	4*h0+3-->h1
1,00099660341	1,08158890887	case 3 with p>=3	11	78	769	5	19	9	32	46	4*h0+3-->h1
1,00043346337	1,08205773804	case 2	12	81	577	1d2p			33	48	4*h0+1-->3*h0+1
1,00057770075	1,08268284361	case 2	13	84	433	1d2p			34	50	4*h0+1-->3*h0+1
1,00076982294	1,08351631770	case 2	14	87	325	1d2p			35	52	4*h0+1-->3*h0+1
0,99692307692	1,08018242134	case 1	15	89	81	0d2p			35	54	4*h0+1-->h0
1,00411522634	1,08462761649	case 2	16	92	61	1d2p			36	56	4*h0+1-->3*h0+1
0,98360655738	1,06684683589	case 1	17	94	15	0d2p			36	58	4*h0+1-->h0
1,02716049383	1,09582292279	case 3 with p>=3	18	102	13	4	1	0	39	63	4*h0+3-->h1
0,92307692308	1,01152885181	case 1	19	104	3	0d2p			39	65	4*h0+1-->h0
1,18518518519	1,19884900955	case N=3	20	111	1				41	70	case N=3

In the first two columns of the previous table, it is shown that the value of the Res satisfies the multiplicative property and the proximity to 1. It is also noted that the maximum Res is reached when N converges to 1 since N'=1 and a=41 and b=70 are the maximum values of the odd and even passages for which Res = (2^70*1)/(3^41*27) = 1.198849. Furthermore, in columns 10 and 11 we note the growth of b with respect to a.

7. Impossibility of non-trivial cycles

If there was a cycle $N \to N' = N$ with $N \neq 1$, con N e N' terms of the sequence included in compound iterations one after the other that are not necessarily successive, we would have:

$$\mathrm{Res}_{\mathrm{cycle}} = \frac{2^{\sum b}}{3^{\sum a}} = 1.$$

But:

- $2^{\sum b} = 3^{\sum a}$ is impossible because 2 and 3 are distinct primes.
- In addition $\sum b > \sum a \Rightarrow \sum b - \sum a \geq number\ of\ iterations$ implies $\mathrm{Res}_{\mathrm{cycle}} > 1$.
- Calculations show that Res is always close to 1 but never equal to 1.

Contradiction, so the only cycle is the trivial one:

$$1 \to 4 \to 2 \to 1.$$

8. Conclusion

The personalized succession used in my works confirms the results of Collatz's canonical succession:

- Every positive integer converges to 1.
- The only possible cycle is the trivial one 1 → 4 → 2 → 1 on the impossibility of having a Res=1.
- The b-a difference always increases with each iteration.

9. References

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