**Existence and Stability Analysis of a Mathematical Model to****Attenuate Head Lice Spread.**

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***Abstract***

Despite effort by public health officials to attenuate head lice, it remains endemic in several part of the globe both in developing and developed countries. This work presents a deterministic mathematical model to attenuate head lice infection. We validate the proposed model by studying it existence and uniqueness solution, computed the basic reproduction number  and the local stability analysis associated with the disease-free equilibrium point. Furthermore, we carefully selected some sensitive parameter and performed numerical simulation on the subdivided population to see their effects for better understanding during decision making.

***Keywords***

Pediculosis, Stability Analysis, Existence Solution, Uniqueness Solution, Basic Reproduction Number and Mathematical Model.

**1. Introduction**

Pediculus humanus capitis normally referred to as head louse is not a disease that threatens human life; however it has remained a source of disturbance to siblings, parents, schools of an infested children and public health officials for several decades. The state of being infested with head lice is called pediculosis capitis. Head lice often leave close to the scalp of the head where they extremely feed on human blood as source of diet and maintain its body temperature. Pediculus humanus capitis also referred to as head lice among other lice such as Phthirus pubis commonly referred to as crab or pubic lice and pediculus humanus corpon’s commonly called clothing or body lice are the only lice out of thousands of ectoparasite in existence that infest human being [1].

There are three stages head lice go through in their life cycle, and they are egg, nymph and adult stage. The eggs (nits) are lay by adult female louse. It takes eight to nine days period to hatch and are often located at the base of the hair shaft where they are usually mistaken for dandruff or hair spray droplets due to their tiny nature [2]. After hatching, it takes seven to twelve days for the nymph to become adult, and it feeds extremely on human blood as it main source of diet and survival. In this stage, the nymph looks like an adult head louse, but it has a pinhead size [2]. The last and not the least stage, is the adult stage. In this stage, the features of adult louse are seen such as legs, actual size and colour. It has six legs, tan to greyish white in colour and its size is about the size of a sesame seed. The survival period for the adult louse on human head where it lives and feeds on human blood as it main source of diet three to ten (3-10) times per day is thirty (30) days [3].

The major places of contacting head lice are in kindergartens, primary and junior secondary schools where head-to-head contact is likely to happen during classroom activities, during play and during school bus riding; thus, transferring same disease to their family members at home via head-to-head contact [4]. This is not far from truth because head nice neither crawl nor hop or fly therefore having head-to-head contact as it only means of contact with an infected individual [5]. Head lice symptoms are tickling feeling of moving object in the hair, itching which often take place in four to six (4-6) weeks period (not excluding first day of infection) after human host develops allergic reaction to head lice saliva, and scratching. There are different kinds of treatment options for pediculosis which are considered effective if the victim is “head lice-free” after receiving the last treatment dose and staying fourteen days without showing signs of head lice [6]. These treatment options are oral treatment, classical topical use of pediculicide such as pyrethis and malathion, and therapeutic wet combing and conditioner, which when used sequel to early detection of head lice infestation in the host population will help put an end to the disease and its spread [7].

In recent years, several mathematical models to study and analyse the dynamics of ectoparasites such as ticks [8], flees [9] and sea lice [10] have emerged. In [11], the authors presented a deterministic stochastic model to study the dynamics of head lice based on stochastic susceptible – infectious – susceptible (SIS). Their analysis shows that if the basic reproduction number is greater than zero, it will take long time for the disease to go on extinction. In addition, [12] presented an endemic infectious model to study the dynamics of head lice infestation. The authors used data collected from recent study in Welsh schools on the prevalence of head lice infestation to gain insight on the dynamics of head lice in a host population using the stationary distribution of the number of infected individuals. The model parameters were obtained using maximum likelihood methods. Their work reveals that if the probability is one, the disease will go on extinction and there wouldn’t be stationary distribution.

Our proposed model is an extension of [13] where the authors have proposed and analysed a continuous time deterministic mathematical model for head lice infestation based on susceptible – infectious – recovered (SIR). In their work, total population was considered fixed, and the disease transmission follows a linear standard incidence. Hence, we formulated a mathematical model for head lice infection by incorporating a new compartment called treated compartment denoted as T(t).

The remaining sections of this work are organized as follows: In section two, we described the proposed model, its parameters, existence and uniqueness of solution, basic properties of the model, model analysis and discussion of numerical simulation and results.

**2. Mathematical Model Formulation**

**2.1 Parameters and State Variable Description**

**Table 1.** Model Parameter and their description

|  |  |  |
| --- | --- | --- |
| **Parameter** | **Description** |  |
|  | Recruitment rate (immigration). |  |
|  |  |  |
|  | Head lice transmission rate |  |
|  |  |  |
|  | Rate at which recovered individual move to susceptible class. |  |
|  |  |  |

 Rate of recovery from the disease

 Treatment rate

 Natural death rate.

**Table 2.** Model State Variable and their description

|  |  |  |
| --- | --- | --- |
| **State Variable** | **Description** |  |
| Susceptible (S) | The class of individuals susceptible at time t. |  |
|  |  |  |
| Infectious (I) | The class of individual infested at time t. |  |
|  |  |  |
| Treated (T) | The class of individuals who are treated at time t. |  |
|  |  |  |

Recovered (R)The class of individuals who recovered

at time t.

Population (N) The total population at time t.

**2.2 Model Flow Diagram**

The compartmental flow diagram below provides a summary insight on the description of head lice disease dynamics in a host population.

**A diagram of a diagram

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**Fig. 1** Model flow diagram.

**2.3 Model Formulation**

In this research work, we extend the model proposed in [17] by incorporating a new compartment called treatment denoted as T(t). The model divide the host population N(t) of a real life system into four compartments namely Susceptible Compartment S(t), Infected Compartment I(t), Treatment Compartment T(t) and Recovered Compartment R(t); where the compartment S(t) refers to individuals who are likely to be infected by the disease, the compartment I(t) refers to individuals who are infected and are able to infect others, the compartment T(t) refers to individuals who are treated of the disease and the compartment R(t) refers to those individuals who have recovered from the disease with temporary immunity. We formulate a mathematical model governing the dynamics of pediculosis spread in a real-life system by making use of the assumptions below:

1. The host population N(t) is varying and mix homogeneously, meaning that all individuals in the population are likely to be infested.
2. There is constant influx into the population at the rate  because of migration.
3. The non-life threatening disease is transferred via head-to-head contact between susceptible individuals and infested individuals at the linear incidence rate  where  is the infection rate and
4. There is no death due to the disease, however there is natural death at the rate for individuals in all compartments and 
5. The infested individuals are treated at the rate  and recovered without permanent immunity.

With the model flow diagram in subsection (2.2) and the assumption in subsection (2.3) in mind, we propose a mathematical model for dynamics of pediculosis as follows:



Where is incidence rate due to migration,is the rate at which recovered individuals return to susceptible compartment due to temporary immunity to the disease, is the infection rate, is the natural death rate,  is the rate at which infested individuals move into the treated compartment, is the rate at which treated individuals recover after treatment and move into the recovered compartment.

Thus, the total population is given as.

**2.4 Existence and Uniqueness solution of the Formulated Model**

Most Real-life problems are being governed by mathematical models [18]. The authenticity and validity of these models are being determined by the existence and uniqueness of the model solution. In this section, we formulate theorem on existence of unique solution of the system in equation (1) – (4) above and provide the proof.

Let the system of model equation (2.1) – (2.4) be



Also considering the system of equations below





Equation (2.7) above is the compact form of equation (2.6).

**Theorem 1:** [15]

Let the region of consideration be , then



Suppose that satisfies the Lipchitz condition then



where k is a positive constant, whenever the pairs  and belong to . Then, there is a constant  such that there exist a unique continuous vector solution of the system (2.5) in the interval  It is sacronsact to note that condition (2.8) is satisfied by the requirement that  are continuous and bounded in  with respect to equation (2.5).

The region of interest is



and a bounded solution of the form



is found in the region , whose partial derivatives satisfies where and are positive constants.

**Theorem 2**: Let the region defined in (2.9) be denoted by  such that equation (2.9) - (2.10) hold. Then, the system of equation (2.5) has a unique solution which is continuous and bounded in the region.

**Proof**

Recall equation (5)



We show that where  are continuous.

For G1, we have

,,  and .

For G2, we have

   and 

For G3, we have

   and 

For G4, we have

   and 

From the computation above, it is obvious all the partial derivate exist, continuous and bounded. Thus, theorem 2 above shows that there exists unique solution of system of equation (2.5) in the region.

**2.5 Basic Properties of the Model**

In this subsection, we study some basic properties of the system of equation in (2.1) – (2.4), namely positivity and boundedness of solution and region of feasible solution. The positivity and boundedness of solution shows the positivity of the model equations while the region of feasible solution is the region in which the solution to the model equation (2.1) – (2.4) make sense biologically.

**2.5.1 Region of Feasible Solution**

Let  be the feasible solution set which is positively invariant set of the model.

**Lemma 1.** The feasible solution set is positive and attract all solution in.

**Proof**

Since the host population is the sum of the four compartments and, that is



Thus









Solving the first order linear differential equation above we have



When the equation above yields



When the equation above becomes 

Thus, the set of feasible solution is positively invariant and attracts all solution in 

**2.5.2 Positivity and Boundedness of Solution**

We have shown that the solution set to the model equation is feasible and attracts all solution in  we now show that the variables of the model equation (2.1) – (2.4) are non-negative.

**Lemma2.** Let be initial state variable values, then the set of solutions  of equation (1) – (4) is positive for all 

**Proof** Let’s consider the system of equation (1)-(4).

From equation (1), if we assume that



Solving the equation above for S(t) we have



When we have



Therefore, 

In like manner, from equation (2),



Solving the equation above for I(t), we have



When  we have



Therefore, 

Also, from equation (3),



Solving the equation above for T(t), yields



When  we have



Therefore, 

Lastly, from equation (4)



Solving the equation above for R(t), yields



When  we have



Therefore, 

Thus, we have established that all state variables are positive 

**3. Model Analysis**

This section analysed qualitatively the system of model equation (2.1) – (2.4) to investigate the equilibrium point and their respective stability analysis to understand the dynamics of pediculosis in a host population.

**3.1 Disease Free Equilibrium (DFE) Point**

Let the disease-free equilibrium (DFE) point be denoted by We obtain the disease-free equilibrium point of the system of model equation (2.1) – (2.4) by setting



Equation (2.1) – (2.4) becomes



In absence of infection (that is I = 0), equation (2.11) yields



Solving equation (2.13) for T, we have



In like manner, solving equation (2.14) for R, yields



Lastly, solving equation (2.12) for S, yields



Therefore, the disease-free equilibrium (DFE) points of the system of model equation (2.1) - (2.4) denoted by  exist and is express as



**3.2 Reproduction Number (Ro)**

The basic reproduction number denoted byis defined as the number of secondary infections produced by a single infectious individual when introduced into a completely susceptible population [16]. The persistence or die out of a disease in a population solely depends on the value of the computed reproduction number. If the value ois less than zero (< 0), implies that the disease will die out, leaving the infectious individuals with no choice but to produce less than one secondary infection. However, if the value of greater than zero (> 0) implies that the disease will persist in the population, thus producing more than one secondary infection and resulting to epidemic.

In most complex epidemic models, basic reproduction number is usually computed using the next generation approach. Nevertheless, since our proposed model has only one infectious compartment, we adopt and use the approach in [21] for computing basic reproduction number.

Let



where

 : Constant of proportionality (=1)

 : The probability of infection, given contact between susceptible and infected.

 : The rate in which the susceptible individual have contact with infected individual.

: The duration of infectiousness.

From equation (2.2) of the system of model equations above,



are rate of removal from the infection compartment. Thus,  becomes the rate of removal from the infectious compartment



Almost every individual in each population is susceptible beside the case where there is index at the beginning of any epidemic [21]. Thus, susceptible individual (S) is approximately unity that is 



At t = 0, we have that



From the equation above, as , decreases.

If the number of infectious individual increases, then epidemic occurs and otherwise 

Thus, equation (2.2) above becomes less than zero,



Recall from disease free equilibrium point that 

Thus, 

**3.3 Local Stability of Disease-Free Equilibrium (DFE) point**

To obtain the local stability analysis of disease-free equilibrium point, we calculate the Jacobian matrix and evaluate it at the disease-free equilibrium point.

Let the system of model equation (1) - (4) be expressed as



The Jacobian matrix is defined as



At the disease-free equilibrium point, equation (2.15) becomes



In other to obtain the local stability of the disease-free equilibrium point, we adopt and use the Routh-Hurwitz condition which state that



With respect to the Jacobian matrix in equation (2.16) above.

If the condition in equation (2.18) is satisfied, then the disease-free equilibrium point is locally asymptotically stable otherwise it is locally asymptotically unstable.

From equation (2.16), we have that



Also, from equation (17), we have that



Recall that



Thus,



Since condition (18) above is satisfied, hence the disease-free equilibrium point  is asymptotically stable provided  thus the disease does not persist (it dies out) within a time interval.

**4. Numerical Simulation**

In this section, we performed a numerical simulation to give more insights to our analytical results. Some of the parameters used are taken from [13] and the remaining parameters were estimated and assumed. Using MATLAB, we performed the numerical simulation of the system of model equation (2.1) – (2.4) subject to the initial conditions given as and different set of parameter values as are captured in the table below at final time 

Table 3. Parameter used in model simulations

|  |  |  |
| --- | --- | --- |
| **Parameter Values** | **Source** |  |
| 0.05 | [13] |  |
|  |  |  |
| 0.002 | [8] |  |
|  |  |  |
| 0.10 | [13] |  |
|  |  |  |

 0.10 Assumed

 0.20 Assumed

 0.05 [13]

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Figure 2Dynamics of Pediculosis infestation over time for different parameter values.

|  |
| --- |
| A graph of different values  AI-generated content may be incorrect. |

Figure 3Effect of different  values on susceptible and infected population over time.

|  |
| --- |
|  |

**A graph of different values

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Figure 4Effect of different values on infected and treated population over time

**A comparison of different values

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|  |
| --- |
|  |
|  |

Figure 5 Effect of different values on infected and treated population over time.

**Discussion of Result**

The parameter analysis, illustrated in Figures 3 through 5, clarifies the dynamics of pediculosis capitis transmission.

Figure 3 demonstrates the impact of transmission rate (β). As β increases, the infested population grows, confirming a direct, linear relationship between the transmission probability and the scale of infestation, while leaving the susceptible population decreasing rapidly within shorter time span.

Similarly, Figure 4 analyzes the loss of recovery or immunity rate (η). An increase in η leads to a decrease in the in the infected class over time, as individuals re-enter the susceptible class. Also, the treated class or population reduces over time, as the values of η increases. This inverse relationship aligns with the model's assumption that recovery does not confer permanent immunity.

Finally, Figure 5 explores the treatment and sensitization rate (). Higher values of  correspond to a more rapid decline in the treated class or population and a concurrent rise in recoveries. This trend indicates that public health initiatives focused on education and treatment can effectively curb the spread of head lice over time.

**Conclusion**

This work present and analysed an epidemic model for transmission dynamics of head lice spread. The model consists of state variables expressed in form of nonlinear ordinary differential equation. The proposed model has in existence one equilibrium (disease free equilibrium) which is locally asymptotically stable provided basic reproduction number  is less than one  We observed that treatment play a significant role in attenuating the spread of head lice. Thus, we strongly recommend treatment as a good control measure to attenuate head lice spread amongst school children.

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**Author’s Biography**

A person with a mustache and hand on chin

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